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# A Pauli-Matrix Formulation of the 3D Maxwell *One-Way* Wave Equation

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## ABSTRACT

Building on the recent work in [K. L. Tsakmakidis, et al., *Nature Communications* 16, 5719 (2025)], we analyze the therein introduced Pauli-matrix first-order formulation pertaining to the Engquist-Majda *one-way* wave problem. We show that the Pauli formulation provides a local doubled matrix factorization whose Hamiltonian is unitarily equivalent to a rotated Weyl Hamiltonian. Because the doubled system contains both longitudinal roots, exact recovery of one scalar one-way branch requires a nonlocal spectral projector. For homogeneous Maxwell fields, the auxiliary branch space should be distinguished from the two-dimensional physical polarization space, leading to a four-component polarization-preserving lift. Numerical snapshots illustrate one-way directional propagation in the selected sector.

## 1 | Introduction

One-way wave equations, that is, equations allowing the propagation of a wave in only, say, the  $+x$  direction for all transverse angles on the  $yz$  plane, and strictly forbidding it in the  $-x$  direction again for all transverse angles, were first studied in the late 1970's and '80s [1–5], long before the arrival of the field of topological physics [7]. A one-way ('half-space') wave operator refers to an operator that, for each transverse Fourier component  $(k_y, k_z)$ , selects only one longitudinal branch  $k_x = \pm\beta$ , where  $\beta = (\omega^2/c^2 - k_y^2 - k_z^2)^{1/2}$ , with the appropriate radiation branch. Perhaps the most well-known and successful example is the pioneering work of Engquist and Majda [1, 2], which triggered a surge of follow-up works. However, up to very recently, all such 'one-way wave equations' have been approximative, i.e., working for only a small range of transverse angles [1–5]. Ref. [6] proposed a Pauli-matrix representation [8] of the square-root structure; below we analyze in detail its relation to branch selection and to Maxwell fields (not considered in [6], which only dealt with the scalar wave equation).

Spinor and Dirac-type reformulations of Maxwell's equations in vacuum date back to Laporte and Uhlenbeck [9], and have been developed extensively since then; see, e.g., Refs. [10, 11]. In these works, the full Maxwell system is written in terms of Dirac operators acting on potentials or field spinors, and the d'Alembertian is factorized in a Lorentz-covariant way. In contrast, the present work starts from the Engquist-Majda half-space one-way problem and studies a local Pauli-matrix representation of its square-root structure.

We first recap the Pauli construction of Ref. [6] and then examine its well-posedness, zero-reflection property, Weyl equivalence and extension to homogeneous Maxwell fields. More general, dispersive or anisotropic cases require additional structure (e.g. matrix-valued 'mass' terms) and will be the subject of future work. We shall show how the Hamiltonian (here meaning the generator of evolution for the first-order hyperbolic system) derived in Ref. [6] can be unitarily mapped to the Weyl Hamiltonian, and how one-way sectors can be represented for homogeneous Maxwell fields. Throughout, we work in a homogeneous, isotropic, source-

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free medium and formulate one-way propagation with respect to a fixed half-space boundary  $x = 0$  in a laboratory frame; the resulting one-way Maxwell system is therefore not Lorentz covariant, i.e., we do not propose a Lorentz-covariant spinor representation of Maxwell's equations. Our starting point is the Engquist–Majda one-way half-space operator, whose defining feature inherently is the selection of a preferred plane and direction; our goal is to formulate a local doubled Pauli representation associated with that one-way operator and to distinguish the auxiliary branch variables from the physical Maxwell polarizations.

## 2 | Scalar One-Way Operator and Pauli Formulation

### 2.1 | Engquist Majda Half-Space Operator

In any homogeneous and isotropic medium, a scalar wavefield  $U$  follows, in three dimensions, the equation  $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0$ . In the electromagnetic context,  $U$  can be taken to be any scalar component of the electric or magnetic field, or an appropriate scalar potential, such as a Debye potential; in a homogeneous isotropic, source-free medium, each such scalar component satisfies the scalar wave equation. Let us define the operator  $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} = L_x^2 + L_y^2 + L_z^2 - \frac{1}{c^2} L_t^2$ , from where the above wave equation takes the form:  $LU = 0$ . In their work [1, 2], Engquist and Majda broke the operator  $L$  into two operators  $L^+$  and  $L^-$ , with the properties  $LU = L^+L^-U = 0$ . These two operators are defined, in three dimensions, as:  $L^+ = L_x + \frac{L_t}{c} \sqrt{1 - \Pi^2}$  and  $L^- = L_x - \frac{L_t}{c} \sqrt{1 - \Pi^2}$ , where we have used  $\Pi = \sqrt{(cL_y/L_t)^2 + (cL_z/L_t)^2}$ . Further, in Ref. [1], it was proved that the act of the operation  $L^-U = 0$  is to propagate  $U$  in the *negative- $x$*  direction (toward  $x = 0$ ) only, *for all angles of incidence*—and similarly for  $L^+U = 0$ , in the *positive- $x$*  direction. The overall approximation in Engquist–Majda's approach lies in the manner in which the *nonlocal* square-root operator is calculated. In particular, Engquist and Majda assumed  $U$  to be a scalar field, thereby in the case where, e.g., a second-order approximation is used, where  $\sqrt{1 - \Pi^2} \approx 1 - \Pi^2/2 + O[\Pi^4]$ , one arrives at:  $L^- \approx L_x - (L_t/c)(1 - \Pi^2/2) = L_x - (L_t/c) + cL_y^2/(2L_t) + cL_z^2/(2L_t)$ . From this last approximation for  $L^-$ , and using the fact that  $L^-U = 0$ , we get the following *approximate* one-way wave equation:

$$\frac{\partial^2 U}{\partial x \partial t} - \frac{1}{c} \frac{\partial^2 U}{\partial t^2} + \frac{c}{2} \frac{\partial^2 U}{\partial y^2} + \frac{c}{2} \frac{\partial^2 U}{\partial z^2} = 0 \quad (1)$$

which describes a wave propagating *only in the negative- $x$  direction* but for a narrow range of incident angles, due to  $\Pi$  having been assumed 'small' ('small' values of  $L_y$  and  $L_z$  per  $L_t$ ).

### 2.2 | Local Pauli Factorization and Branch Selection

Ref. [6] proposed a Pauli-matrix doubled representation of the square-root structure; below we clarify its relation to branch selection and to Maxwell fields. Note that we do not introduce a new Lorentz-covariant spinor form of Maxwell's equations, but an exact Pauli-matrix factorization of the **Engquist–Majda one-**

**way operator** and its extension to the Maxwell tangential field (see later on) in a homogeneous isotropic half-space. In particular, exploiting the fact that  $\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} I$ , all cross terms in the expansion  $(L_y^2 + L_z^2 - L_t^2)^2$  mutually cancel, and as a result the following equality is obtained *exactly* (see also Figure 1):

$$Q := \sigma_x L_y + \sigma_y L_z + i\sigma_z L_t, \quad Q^2 = (L_y^2 + L_z^2 - L_t^2) I_2 \quad (2)$$

Thus  $Q$  is a local matrix square root of  $(L_y^2 + L_z^2 - L_t^2)I_2$ . It should not necessarily be identified with the scalar principal square root  $\sqrt{L_y^2 + L_z^2 - L_t^2} I_2$  unless a spectral branch of  $Q$  has been selected. To clearly see the logic behind the derivation of Equation (2), consider an ansatz  $Q = AL_y + BL_z + CL_t$  with  $2 \times 2$  matrices  $A, B, C$ . Requiring  $Q^2 = (L_y^2 + L_z^2 - L_t^2)I_2$  for all functions, forces  $A^2 = B^2 = C^2 = I_2$ ,  $C^2 = -I_2$ , and  $AB + BA = AC + CA = BC + CB = 0$ . Up to similarity transformations, the unique such matrices are the Pauli matrices with  $A = \sigma_x$ ,  $B = \sigma_y$ ,  $C = i\sigma_z$ , yielding Equation (2). Squaring the right-hand side of Equation (2), leads to all multiplication cross-terms cancelling because of the above properties of the  $\sigma$  matrices. If one formally replaces the scalar square-root operator by the matrix square root  $Q$ , one obtains the local doubled first-order system  $(L_x iQ)\psi = 0$ . This system is equivalent to the scalar one-way branch on a projected subspace of the doubled field. The required projectors are  $\Pi_{\pm} = \frac{1}{2} (I_2 \pm \frac{Q}{\sqrt{Q^2}}) = \frac{1}{2} (I_2 \pm \frac{Q}{\sqrt{A}})$ , where  $A = L_y^2 + L_z^2 - L_t^2$ . This gives rise to the local doubled Pauli/Weyl-type first-order system:

$$i\partial_t \psi = (\sigma_y D_y - \sigma_x D_z + \sigma_z D_x) \psi \quad (3)$$

For a Fourier component  $\exp[i(k_x x + k_y y + k_z z - \omega t)]$ , we can define  $q(\omega, \mathbf{k}_{\perp}) = i\sigma_x k_y + i\sigma_y k_z + \frac{\omega}{c} \sigma_z$ ,  $q^2 = \beta^2 I_2$ , where  $\beta = \sqrt{\frac{(\omega + i0^+)^2}{c^2} - k_y^2 - k_z^2}$ . Equation (3) is therefore a local doubled system containing the two longitudinal roots. The corresponding spectral projectors are  $\Pi_{\pm} = \frac{1}{2} (I_2 \pm \frac{q}{\beta})$ . These projectors are defined for  $\beta \neq 0$ . Exact grazing ( $\beta = 0$ ), where the longitudinal roots coalesce, is excluded from this decomposition and must be treated by a separate limiting prescription. They are applied by Fourier transforming in  $t, y$  and  $z$ , multiplying each spectral component by  $\Pi_+$  or  $\Pi_-$ , and inverse transforming. Because  $\beta$  contains the square root, this exact branch-selection operation is nonlocal and is not a fully local replacement for the Engquist–Majda operator. Without projection, the field evolves as  $\hat{\psi}(x) = e^{i\beta(x-x_0)} \Pi_+ \hat{\psi}(x_0) + e^{-i\beta(x-x_0)} \Pi_- \hat{\psi}(x_0)$ , so generic data propagate as a superposition of the two longitudinal branches. The conceptual advantage of the Pauli representation is instead that it converts the doubled dispersion relation into a local first-order matrix system, makes the two branches and their projectors explicit, and exposes its Weyl structure while retaining an exact all-angle description.

## 3 | Unitary Equivalence to the Weyl Hamiltonian

We now show that the Hamiltonian of the doubled Pauli system in Equation (3) is unitarily equivalent to a rotated Weyl Hamiltonian. To that end, we note that the Hamiltonian associated with Equation (3) can be written as  $H_{OW} = \sigma \cdot \alpha$ , where  $\sigma =$

$$(L_y^2 + L_z^2 - L_t^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} iL_t & L_y - iL_z \\ L_y + iL_z & -iL_t \end{pmatrix} \begin{pmatrix} iL_t & L_y - iL_z \\ L_y + iL_z & -iL_t \end{pmatrix}$$

$$\sqrt{(L_y^2 + L_z^2 - L_t^2)I} = L_y \sigma_x + L_z \sigma_y + iL_t \sigma_z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

**FIGURE 1** | The Pauli  $\sigma$ -matrices construction gives a local matrix operator  $Q$  (see main text), satisfying  $Q^2 = (L_y^2 + L_z^2 - L_t^2)I_2$ . Thus  $Q$  is a matrix square root of the scalar square-root argument multiplied by  $I_2$ . It is not identical to the scalar principal square root; selecting one one-way branch requires a spectral projector involving the scalar square-root operator.

$(\sigma_x, \sigma_y, \sigma_z)$  and  $\mathbf{a} = (-k_z, k_y, k_x)^T$ . Now let us define the  $3 \times 3$  matrix:

$$R = R_y \left( -\frac{\pi}{2} \right) = \begin{pmatrix} \cos\left(-\frac{\pi}{2}\right) & 0 & -\sin\left(-\frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(-\frac{\pi}{2}\right) & 0 & \cos\left(-\frac{\pi}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \text{SO}(3) \quad (4)$$

From this, assuming that  $\mathbf{k} = (k_x, k_y, k_z)$ , we immediately get:  $R\mathbf{k} = (-k_z, k_y, k_x)^T = \mathbf{a}$ , i.e.  $H_{\text{OW}} = \boldsymbol{\sigma} \cdot (R\mathbf{k})$ . Further, concerning the wavefield  $U$  itself, knowing that a rotation by an angle  $\theta$  about the  $y$ -axis corresponds to  $W(\theta) = \exp(-\frac{i}{2}\theta\sigma_y)$ , we use the fact that [8]  $W\sigma_x W^\dagger = \sigma_x \cos\theta + \sigma_z \sin\theta$ ,  $W\sigma_y W^\dagger = \sigma_y$ ,  $W\sigma_z W^\dagger = \sigma_z \cos\theta - \sigma_x \sin\theta$ , and for  $\theta = \pi/2$ , we have:  $W\sigma_x W^\dagger = \sigma_z$ ,  $W\sigma_y W^\dagger = \sigma_y$ , and  $W\sigma_z W^\dagger = -\sigma_x$ , i.e. exactly analogous to the operation of  $R_y(-\frac{\pi}{2})$ . Thus, we obtain the following, exact, unitary equivalence between the one-way wave Hamiltonian  $H_{\text{OW}}$  and the Weyl Hamiltonian  $H_W = \boldsymbol{\sigma} \cdot \mathbf{k}$ :

$$H_{\text{OW}} = \boldsymbol{\sigma} \cdot (R\mathbf{k}) = W(\boldsymbol{\sigma} \cdot \mathbf{k})W^\dagger = W H_W W^\dagger. \quad (5)$$

Equation (5) shows that the complete doubled Pauli Hamiltonian is unitarily equivalent to a Weyl Hamiltonian. Its Chern number is associated with an energy-band projector of the full Hamiltonian over a closed momentum-space surface surrounding the Weyl point [6]. This energy-band projection is distinct from the longitudinal projector  $\Pi_\pm$ , which selects one of the two roots  $k_x = \pm\beta$  at fixed frequency and transverse wavevector. Consequently, longitudinal branch projection does not itself create a Chern number or establish topologically protected transport. The Maxwell lift introduced below gives two polarization-degenerate copies of the auxiliary Weyl structure, not a topological photonic band structure in the homogeneous material.

## 4 | Well Posedness and Zero Reflection

### 4.1 | Well Posedness of the Doubled System

Since a two-component wavefield (rather than a scalar field) is necessary for the exact one-way wave equation, where we now have two coupled equations, it is only natural to ask

whether the so-resulting equation and problem are well-posed [3]. Fortunately, it turns out that they are, for the reasons concisely outlined in the following. First of all, we note that the coefficients on the right-hand side of Equation (3), i.e. the Pauli matrices, are constant Hermitian  $2 \times 2$  matrices, i.e., Equation (3) is a symmetric-hyperbolic system of equations with constant coefficients. If we define the energy density as  $\rho = \psi^\dagger \psi$ , we may straightforwardly show that  $\partial_t(\psi^\dagger \psi) + \partial_x(\psi^\dagger \sigma_x \psi) + \partial_y(\psi^\dagger \sigma_y \psi) - \partial_z(\psi^\dagger \sigma_z \psi) = 0$ , i.e. in  $R^3$  the  $L^2$  norm  $\|\psi(\cdot, t)\|_2$  is conserved. Further, assuming a plane-wave solution of the form  $\psi(\mathbf{r}, t) = \mathbf{u}(\mathbf{k})e^{i(\mathbf{k}\mathbf{r} - \omega t)}$ , we have from Equation (3) that  $\omega \mathbf{u}(\mathbf{k}) = (\sigma_y k_y - \sigma_x k_z + \sigma_z k_x) \mathbf{u}(\mathbf{k})$ , and because  $\boldsymbol{\sigma} \cdot \mathbf{a}$  has eigenvalues  $\pm |\mathbf{a}|$ , and we look for positive-frequency excitations ( $\omega > 0$ ) as in standard scattering theory, it is  $\omega = +|\mathbf{k}|$ ; hence, the group velocity becomes  $\mathbf{v}_g = \nabla_{\mathbf{k}} \omega = +\mathbf{k}/|\mathbf{k}|$ , i.e.  $|\mathbf{v}_g| = 1$  ( $= c$  if we restore units), i.e. it has a *finite* magnitude, it is non-dispersive, and there are no superluminal ‘tails’ (precursors).

### 4.2 | Zero Reflection in the Selected Branch

We may also analytically calculate the reflection coefficient  $R$  for an incident field with tangential wavenumber  $\mathbf{k}_\perp = (k_y, k_z)$  and  $|\mathbf{k}_\perp| < \omega/c$ . Indeed, let us assume that  $\beta = ((\omega/c)^2 - |\mathbf{k}_\perp|^2)^{1/2}$  and that the total field in the  $x > 0$  region is  $\psi(x, y, z, t) = u_{\text{inc}} e^{i(-\beta x + k_y y + k_z z - \omega t)} + R u_{\text{ref}} e^{i(+\beta x + k_y y + k_z z - \omega t)}$ . Then, for  $\omega > 0$ , it is  $L^-(\omega, k_y, k_z) = i(k_x + \beta)$ , and therefore  $L^-\psi|_{x=0} = i((-\beta) + \beta)u_{\text{inc}} + i((+\beta) + \beta)Ru_{\text{ref}} = i(2\beta)Ru_{\text{ref}}$ , but because  $L^-\psi|_{x=0} = 0$ , we immediately get  $R = 0$ , for all  $\omega, k_y, k_z$  with  $|\mathbf{k}_\perp| < \omega/c$ . Thus, the scalar Engquist-Majda projected branch gives  $R = 0$ , and the Pauli system reproduces that result after the same branch projection is imposed, i.e., the wavefield is smoothly ‘absorbed’ by the boundary and propagates solely in the  $-x$  direction (Figure 3 illustrates the complementary  $+x$  sector). Note that the same is true for evanescent waves, too (where  $|\mathbf{k}_\perp| > \omega/c$ ), because in that case we simply have  $\beta = i\alpha$ , where  $\alpha > 0$ , giving again  $R = 0$  from the above  $L^-\psi|_{x=0} = 0$  condition. From these considerations, we may conclude that the doubled local Pauli/Weyl Cauchy system is well-posed, and that the one-way projected sector inherits well-posedness after the projection/radiation branch is specified. At exact grazing incidence,  $\beta = 0$ , the incident/reflected decomposition and the division by  $2\beta$  used above are singular; this isolated case is excluded from the formulas as written and requires a separately prescribed limiting treatment.

## 5 | Extension to homogeneous Maxwell fields

### 5.1 | Branch and Physical Polarization Spaces

We shall now extend these ideas to the full-Maxwell regime [12]. In the full Maxwell case, each tangential transverse-electric (TE) and transverse-magnetic (TM) amplitude  $E_s, E_p$  also satisfies the scalar wave equation in the homogeneous isotropic medium, so the scalar Engquist-Majda operator and its Pauli factorization can be applied component-wise to these amplitudes. To that end, let us first define the tangential TE/TM unit vectors on the  $x = \text{const.}$  plane; if perpendicularly to the  $x$ -plane, it is  $\mathbf{n} = \hat{\mathbf{x}}$ , and tangentially  $\hat{\mathbf{k}}_{\perp} = \mathbf{k}_{\perp}/|\mathbf{k}_{\perp}|$  (for  $\mathbf{k}_{\perp} \neq 0$ ), then the tangential TE vector is  $\mathbf{s} = \mathbf{n} \times \hat{\mathbf{k}}_{\perp}$ , while the tangential TM vector is  $\mathbf{p} = \hat{\mathbf{k}}_{\perp}$ . We may then decompose the tangential electric field as  $\mathbf{E}_t = E_s \mathbf{s} + E_p \mathbf{p}$ . The auxiliary Pauli components span a branch space, whereas the TE and TM amplitudes span an independent physical-polarization space:  $\mathcal{H}_b \cong \mathbb{C}^2$ ,  $\mathcal{H}_{\text{pol}} = \text{span}\{\text{TE, TM}\} \cong \mathbb{C}^2$ ,  $\Psi \in \mathcal{H}_b \otimes \mathcal{H}_{\text{pol}} \cong \mathbb{C}^4$ . Here,  $I_{\text{pol}}$  denotes the  $2 \times 2$  identity on  $\mathcal{H}_{\text{pol}}$ . Define  $P = \sigma_x c \partial_y + \sigma_y c \partial_z + i \sigma_z \partial_t$ . Then:

$$\left( \frac{i}{c} P \otimes I_{\text{pol}} \right)^2 = \left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla_{\perp}^2 \right) I_4 \quad (6)$$

up to the stated Fourier-sign convention. Thus  $iP/c$  is a local matrix square root of the scalar square-root argument. It is not identical to the scalar principal square-root operator unless a spectral branch is selected. Equation (6) gives a local matrix square root of the scalar square-root argument. It does not by itself replace the nonlocal half-space Fourier multiplier because the exact one-way branch still requires spectral projection. With that qualification, one can write the following local doubled first-order system:

$$\left[ \frac{\partial}{\partial x} I_4 - \frac{i}{c} (P \otimes I_{\text{pol}}) \right] \Psi = 0. \quad (7)$$

Branch selection is performed by  $P_{\pm} = \Pi_{\pm} \otimes I_{\text{pol}}$ ,  $\text{rank } P_{\pm} = 2$ . Thus the Pauli matrices act only in the auxiliary branch space. After one longitudinal branch is selected, the rank-two physical-polarization space remains, preserving the two independent Maxwell polarizations. For a selected branch, the field may be written as  $\Psi_{\pm} = \chi_{\pm} \otimes \begin{pmatrix} E_s \\ E_p \end{pmatrix}$ ,  $q \chi_{\pm} = \pm \beta \chi_{\pm}$ , where  $E_s$  and  $E_p$  remain independent. The remaining electric- and magnetic-field components then follow from the transversality and impedance relations given below. These projectors are defined for  $\beta \neq 0$ . Exact grazing ( $\beta = 0$ ), where the longitudinal roots coalesce, is excluded from this decomposition and must be treated by a separate limiting prescription. Note that  $\Psi$  is not a Lorentz spinor, and Equation (7) is not Lorentz covariant, as expected for any fixed-boundary absorbing/one-way formulation. Indeed, because one-way propagation is defined with respect to a specific plane  $x = 0$  and a specific spatial direction (the  $x$ -axis), the full Lorentz symmetry of *vacuum* Maxwell equations is necessarily broken: a Lorentz transformation does not preserve both the homogeneous isotropic medium and a fixed one-way boundary condition. Equation (7) is therefore formulated in a particular laboratory frame, just as the original Engquist-Majda one-way equations and absorbing boundary conditions are.

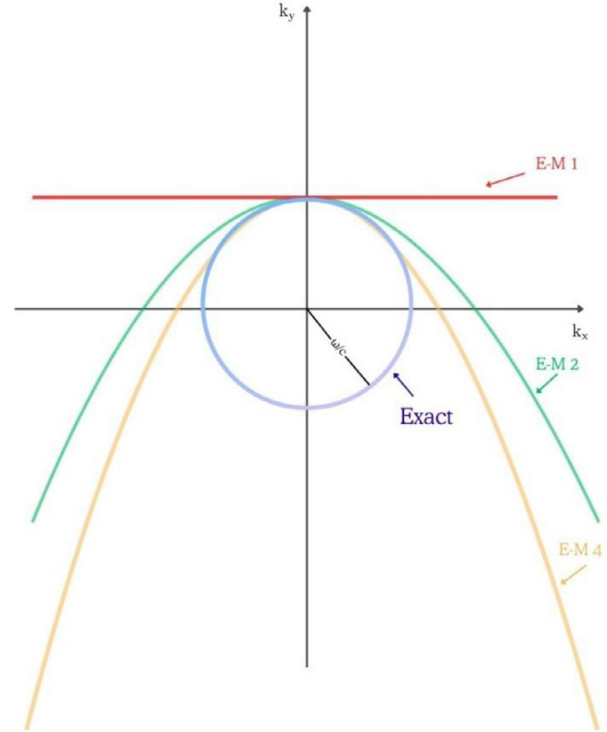


**FIGURE 2** | The dispersion equations of the various Engquist-Majda (E-M) approximate one-way wave equations (see Figure 1 in Ref. [2]), and of the present Pauli system reproducing the exact, free-space dispersion,  $k_x^2 + k_y^2 = (\frac{\omega}{c})^2$ . One-way behavior requires selecting one sign of  $k_x$ .

### 5.2 | Branch Selection and Maxwell Field Reconstruction

Equation (7) contains both longitudinal roots. The  $+x$  and  $-x$  one-way sectors are obtained by imposing  $P_+ \Psi = \Psi$  ( $k_x = +\beta$ ) and  $P_- \Psi = \Psi$  ( $k_x = -\beta$ ), respectively. Equivalently, the latter may be written as  $[\partial_x I_4 + (i/c)(P \otimes I_{\text{pol}})]\Psi = 0$  while selecting the  $P_+$  eigenspace; the operator sign and branch projector must therefore be chosen consistently. For each Fourier component, let  $k_x = \sigma\beta$ ,  $\sigma = \pm 1$ ,  $\kappa = |\mathbf{k}_{\perp}|$ ,  $\mathbf{k} = \sigma\beta \mathbf{n} + \kappa \mathbf{p}$ , and  $\mathbf{E} = E_x \mathbf{n} + E_p \mathbf{p} + E_s \mathbf{s}$ , so that  $\mathbf{E}_t = E_s \mathbf{s} + E_p \mathbf{p}$ . For  $\beta \neq 0$ ,  $\mathbf{k} \cdot \mathbf{E} = 0$  and  $\mathbf{H} = (1/\omega\mu) \mathbf{k} \times \mathbf{E}$  give

$$\begin{aligned} E_x &= -\frac{\kappa}{\sigma\beta} E_p, & H_x &= \frac{\kappa}{\omega\mu} E_s, \\ H_s &= \frac{\sigma\omega\epsilon}{\beta} E_p, & H_p &= -\frac{\sigma\beta}{\omega\mu} E_s \end{aligned}$$

These are the explicit field-reconstruction and TE/TM impedance relations for the selected branch ( $\sigma = +1$  for  $+x$  and  $\sigma = -1$  for  $-x$ ).

The normal components above are consistent with the divergence conditions  $\nabla \cdot (\epsilon \mathbf{E}) = \nabla \cdot (\mu \mathbf{H}) = 0$ . From Equations (6) and (7) it is straightforward to show that, for the isotropic case, the dispersion relation pertaining to Equation (7) becomes  $k_x^2 + k_y^2 + k_z^2 = (\frac{\omega}{c})^2$ , matching exactly the free-space dispersion equation—as it should (see Figure 2).

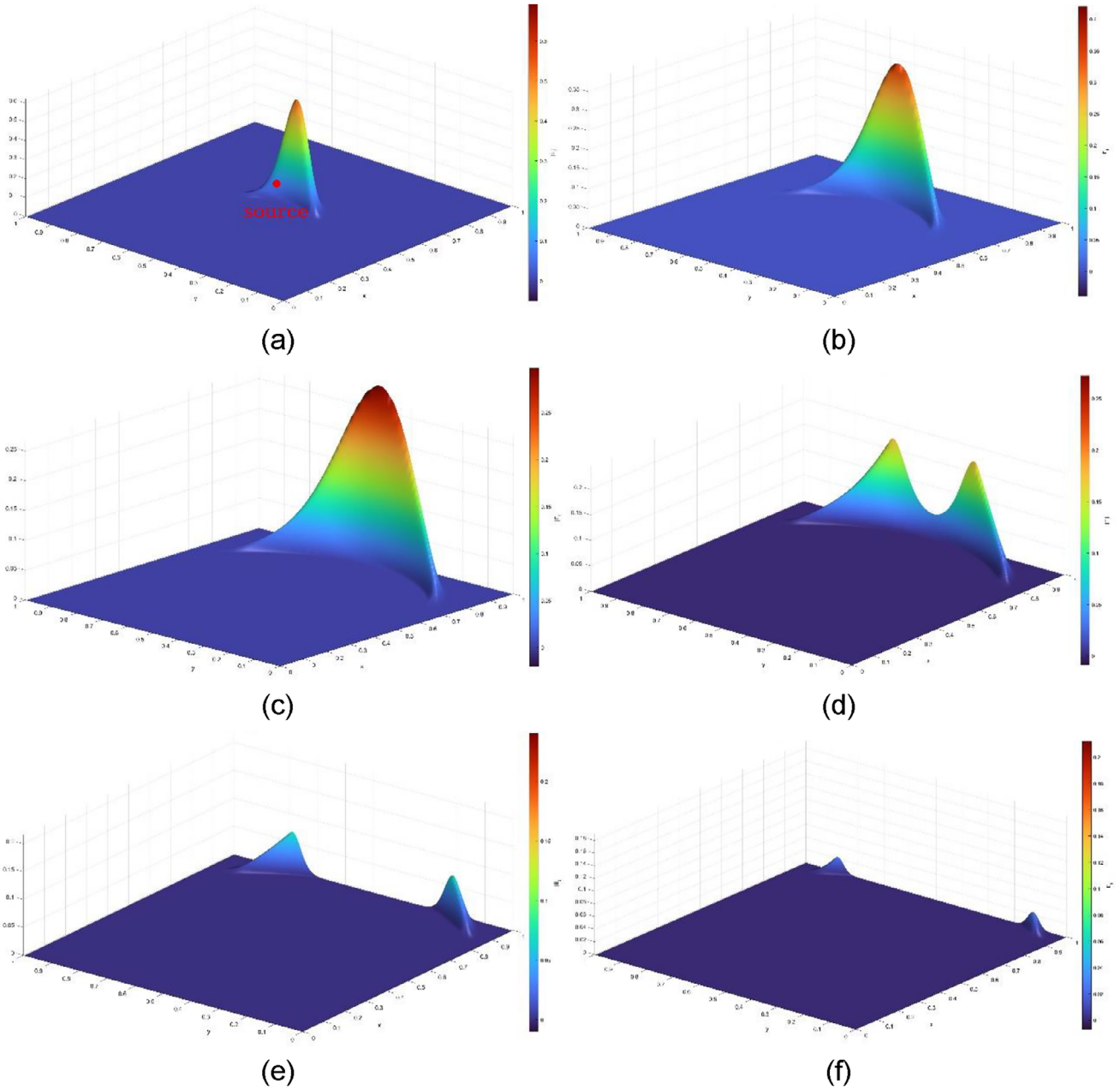


**FIGURE 3** | (a–f) Successive snapshots from the time-domain numerical solution of Equation (7) of the main text, for the  $k_z = 0$ ,  $\mathcal{P}_+$ -projected branch, reconstructed as  $|\mathbf{E}_t|$ , with a source placed at the center of a free-space domain. The half-cylindrical wave illustrates propagation in the selected  $+x$  sector and its arrival at the right boundary. The all-angle zero-reflection result is established analytically in the text; the snapshots, here, numerically corroborate this result.

## 6 | Numerical Illustration

To illustrate the solution of Equation (7), Figure 3 shows time-domain snapshots for the two-dimensional case  $k_z = 0$ . For each spectral component  $(\omega, k_y)$ , the source spectrum is projected with  $\mathcal{P}_+(\omega, k_y, 0)$ , retaining  $k_x = +\beta$ , and is then inverse transformed. The projected four-component state is represented as  $\hat{\Psi}_+ = \chi_+ \otimes \mathbf{e}$ , with  $\mathbf{e} = (E_s, E_p)^T$ ; the entries of  $\chi_+$  are auxiliary branch components and are not identified with the physical TE/TM amplitudes. For normalized  $\chi_+$ , the physical-polarization amplitudes are reconstructed spectrally as  $\mathbf{e} = (\chi_+^\dagger \otimes I_{\text{pol}}) \hat{\Psi}_+$ , and the tangential electric field is  $\mathbf{E}_t = E_s \mathbf{s} + E_p \mathbf{p}$ . In the two-dimensional geometry shown,  $\mathbf{p} = \hat{\mathbf{y}}$  and  $\mathbf{s} = \hat{\mathbf{z}}$ , so  $E_y = E_p$ ,

$E_z = E_s$ , and the plotted quantity is  $|\mathbf{E}_t| = (|E_y|^2 + |E_z|^2)^{1/2}$ . The snapshots illustrate a half-cylindrical wave propagating in the selected  $+x$  sector and reaching the right boundary. The all-angle zero-reflection result for  $\beta \neq 0$  follows from the analytical branch calculation above, i.e., not from the finite set of snapshots [6, 7].

## 7 | Conclusions and Outlook

In conclusion, we have analyzed the Pauli-matrix doubled first-order system associated with the scalar half-space one-way problem and shown that its Hamiltonian is unitarily equivalent

to a rotated Weyl Hamiltonian. We have also discussed how this doubled representation can be applied to homogeneous isotropic Maxwell fields in TE/TM angular-spectrum variables. The Pauli construction gives a local representation of the doubled dispersion relation, whereas exact selection of one longitudinal branch remains nonlocal. In the homogeneous Maxwell lift, auxiliary branch and physical-polarization spaces are treated separately so that both transverse polarizations survive branch selection. The Weyl Chern number belongs to the complete doubled Hamiltonian and does not, by itself, imply topologically protected propagation. The tensor-product formulation above is exact for homogeneous isotropic media. In an inhomogeneous medium, spatially varying  $\epsilon$  and  $\mu$  do not commute with the derivative operators and generally couple the two physical polarizations, so the present constant-coefficient  $2 \times 2$  Pauli factor cannot be transferred unchanged. A general extension must begin from

$$\text{the four-component tangential Maxwell field } F = \begin{pmatrix} E_y \\ E_z \\ H_y \\ H_z \end{pmatrix}, \quad \frac{\partial F}{\partial x} =$$

$\mathcal{M}(\mathbf{r}, D_{\perp}; \omega)F$ , and construct its two-dimensional forward and backward subspaces. The present result should therefore be regarded as the homogeneous reference case for such a future extension.

#### Author Contributions

**Maria Bazini:** investigation, writing – original draft, validation, visualization, writing – review and editing. **Elena Psychogiou:** investigation, methodology, validation. **Maria Barlou:** conceptualization, investigation, methodology, validation. **Konstantinos G. Baskourellos:** conceptualization, investigation, methodology, validation. **Kosmas L. Tsakmakidis:** conceptualization, writing – original draft, methodology, writing – review and editing, project administration, resources, supervision, funding acquisition.

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#### Conflicts of Interest

The authors declare no conflicts of interest.

#### Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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